

Directional geodesics in a smooth Finsler shell

Convexity, completeness, and exact scattering relations

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Abstract

A smooth inward directional correction defines a nonreversible Finsler metric on Euclidean space. The correction has compact support in a spatial annulus and vanishes to every order in directions tangent to concentric spheres. An explicit Hessian calculation gives a sharp positive-amplitude threshold of approximately 1.917643. The bounded annular model is geodesically complete. In two dimensions, conserved energy and angular momentum determine each incoming branch through one scalar equation. The closest-approach radius equals the Euclidean impact parameter, while the excess scattering angle is positive for rays that intersect the active shell. The outgoing branch is exactly Euclidean. Radial geodesics cross the shell without reflection at any positive radius. Analytical quadrature and direct Euler–Lagrange integration agree in the supplied numerical example. These results describe a prescribed positive definite spatial geometry with an explicit shell profile.

1 Introduction

This study defines an inward directional deformation of a Euclidean length function. A smooth annular profile controls its spatial support. One geodesic Lagrangian determines the resulting trajectories. The model admits exact convexity and scattering results, with independent numerical verification.

The construction belongs to an established metric class. Yu and Zhu define general (α, β) -metrics of the form $F = \alpha\phi(x, \beta/\alpha)$ [1, Definition 3.1]. Their vertical convexity criteria provide a standard reference for the calculation below. Mo and Zhou develop the broader calculus of spherically symmetric Finsler metrics [2]. This study specializes those geometric ideas to a smooth directional response and derives its trajectory properties.

The background geometry and shell profile are prescribed. The length scale and amplitude are independent parameters of the model. The proofs establish its regularity, completeness, and scattering behavior under the stated assumptions.

2 A fully specified model

Let $x \in \mathbb{R}^d$, with $d \geq 2$, and let $y \in T_x\mathbb{R}^d$. Write $r = |x|$ and $\alpha = |y|$. For $r > 0$ and $y \neq 0$, define

$$s = \frac{x \cdot y}{r\alpha}, \quad -1 \leq s \leq 1. \quad (2.1)$$

Negative s denotes inward motion. Positive s denotes outward motion. The value $s = 0$ denotes a tangent direction to a sphere about the origin.

Define the smooth directional response

$$q(s) = \begin{cases} \exp(-1/s^2), & s < 0, \\ 0, & s \geq 0. \end{cases} \quad (2.2)$$

Let $A : [0, \infty) \rightarrow [0, 1]$ be smooth. Assume $A(r) = 0$ for $r \leq r_-$ and $r \geq r_+$, where $0 < r_- < r_+ < \infty$. Set

$$\phi(r, s) = 1 + A(r)q(s), \quad F(x, y) = \alpha\phi(r, s) \quad (r > 0, y \neq 0). \quad (2.3)$$

Set $F(0, y) = |y|$ and $F(x, 0) = 0$. The formula equals $|y|$ on a neighborhood of the origin. It therefore avoids the coordinate singularity of the radial one-form dr .

On $r > 0$, this is a general (α, β) -metric with $\beta = dr(y)$. Its ratio β/α equals s . The direction variable and amplitude are dimensionless. If an affine parameter has units, F has the same units as α . The integral $\int F d\tau$ has units of length. The affine parameter has no physical time calibration.

The numerical example uses a length scale $\ell > 0$ and $A(r) = \epsilon\chi(r/\ell)$, with $0 \leq \epsilon \leq 1$. Its profile is

$$\chi(\rho) = \begin{cases} \exp\left(1 - \frac{1}{1 - (\rho - 2)^2}\right), & 1 < \rho < 3, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

The profile reaches one at $\rho = 2$. The length scale and amplitude are prescribed independently.

3 Regularity and strong convexity

3.1 Smoothness and positive homogeneity

Every derivative of $\exp(-1/s^2)$ tends to zero as $s \rightarrow 0^-$. Thus q is smooth on $\mathbb{R}$ and flat at zero. Here, flat means that every derivative vanishes. The annular support removes the separate issue at $x = 0$. Consequently, F is smooth on the slit tangent bundle $T\mathbb{R}^d \setminus \{0\}$.

For $t > 0$, the substitution $y \mapsto ty$ preserves s and gives

$$F(x, ty) = tF(x, y). \quad (3.1)$$

This positive homogeneity makes curve length independent of an orientation-preserving change of parameter. For $|s| \leq 1$, the elementary bound $0 \leq q \leq e^{-1}$ also gives

$$|y| \leq F(x, y) \leq (1 + e^{-1})|y|. \quad (3.2)$$

Strong convexity requires a separate Hessian calculation.

3.2 The fundamental tensor

Fix x with $r > 0$. Define

$$u = \frac{y}{\alpha}, \quad n = \frac{x}{r}, \quad w = n - su, \quad |w|^2 = 1 - s^2. \quad (3.3)$$

Primes on ϕ denote derivatives with respect to s at fixed r . Introduce

$$B = \phi - s\phi', \quad C = B + (1 - s^2)\phi''. \quad (3.4)$$

Direct differentiation gives

$$d_y F = \phi u + \phi' w, \quad (3.5)$$

$$\alpha \text{Hess}_y F = B(I - u \otimes u) + \phi'' w \otimes w. \quad (3.6)$$

For a test vector $v = au + v_\perp$, with $v_\perp \perp u$, the fundamental tensor is

$$\begin{aligned} g_y^F(v, v) &= \frac{1}{2} \text{Hess}_y(F^2)(v, v) \\ &= [\phi a + \phi' \langle w, v_\perp \rangle]^2 + \phi [B|v_\perp|^2 + \phi'' \langle w, v_\perp \rangle^2]. \end{aligned} \quad (3.7)$$

The tangential bracket has coefficient C along w and coefficient B in its orthogonal complement. Thus $\phi > 0$, $B > 0$, and $C > 0$ imply positive definiteness. At $s = \pm 1$, the vector w vanishes, and the same formula remains regular. These factors agree with the standard pointwise (α, β) criteria [1, Propositions 3.2–3.3].

3.3 An explicit sharp amplitude threshold

Theorem 3.1 (Strong convexity threshold). *Allow any nonnegative pointwise amplitude A . At $r > 0$, the norm in (2.3) is strongly convex if and only if $A < A_*$. Here*

$$A_* = -\frac{1}{H(z_*)} = 1.9176430337\dots, \quad H(z) = e^{-z}(4z^3 - 10z^2 + 4z + 1), \quad (3.8)$$

and z_* is the unique root in $(1, 3/2)$ of

$$-4z^3 + 22z^2 - 24z + 3 = 0. \quad (3.9)$$

In particular, the selected range $0 \leq A \leq 1$ defines a regular Finsler metric.

Proof. For $s \geq 0$, the fundamental tensor equals the Euclidean tensor. For $s < 0$, put $z = 1/s^2 \geq 1$. Then

$$q' = 2s^{-3}e^{-1/s^2}, \quad q'' = e^{-1/s^2}(4s^{-6} - 6s^{-4}), \quad (3.10)$$

$$B = 1 + Af(z), \quad f(z) = e^{-z}(1 - 2z), \quad (3.11)$$

$$C = 1 + AH(z). \quad (3.12)$$

The minimum of f is $-2e^{-3/2}$ at $z = 3/2$. For $z \geq 3/2$, the derivative q'' is nonnegative, so $H(z) \geq f(z)$. On $[1, 3/2]$, write $H'(z) = e^{-z}R(z)$, with R equal to the cubic in (3.9). Its derivative satisfies

$$R'(z) = -4(3z - 2)(z - 3) > 0. \quad (3.13)$$

The values $R(1) = -3$ and $R(3/2) = 3$ give exactly one root z_* in this interval. The minimum $H(z_*)$ is strictly below $H(3/2) = -2e^{-3/2}$. It is therefore the global minimum on $[1, \infty)$.

Numerically, $z_* = 1.2850272288\dots$ and $H(z_*) = -0.5214734872\dots$. For $0 \leq A < A_*$, both B and C are positive. Equation (3.7) proves strong convexity.

At $A = A_*$, the coefficient C vanishes at $s_* = -1/\sqrt{z_*}$. Choose $v_\perp \parallel w$ and $a = -\phi' \langle w, v_\perp \rangle / \phi$. The squared term in (3.7) vanishes, so this nonzero vector has zero fundamental-tensor norm. For $A > A_*$, the same construction gives a negative value. This proves sharpness in every dimension $d \geq 2$. $\square$

For $A \leq 1$, the uniform factor bounds are

$$B \geq 1 - 2e^{-3/2} = 0.5537396797\dots, \quad C \geq 1 - 1/A_* = 0.4785265128\dots \quad (3.14)$$

These analytical bounds cover every direction. The numerical eigenvalue scan in Section 7 is a diagnostic, not the convexity proof.

3.4 Directional asymmetry

Where $A(r) > 0$, inward directions receive a positive correction and outward directions receive none. The pointwise reversibility constant is exactly

$$\sup_{y \neq 0} \frac{F(x, -y)}{F(x, y)} = 1 + \frac{A(r)}{e}. \quad (3.15)$$

The maximum occurs for an outward radial vector whose reverse is inward radial. This is an asymmetry of the norm. The trajectory calculation below establishes its effect on geodesics for the prescribed shell.

4 Complete geodesics

The geodesic Lagrangian and its equations are

$$L(x, \dot{x}) = \frac{1}{2}F(x, \dot{x})^2, \quad \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0. \quad (4.1)$$

This action governs motion in the specified geometry. The background and profile remain prescribed in this variational problem. Positive definiteness of $L_{\dot{x}\dot{x}}$ gives a smooth geodesic spray on the slit tangent bundle.

The Lagrangian has no explicit parameter dependence and is homogeneous of degree two in velocity. Its conserved energy is therefore L , and each nonconstant affine geodesic has constant Finsler speed $F = v_F > 0$.

Proposition 4.1 (Geodesic completeness). *Every nonconstant affine geodesic of the annular model extends to all $\tau \in \mathbb{R}$.*

Proof. Equation (3.2) gives

$$\frac{v_F}{1 + e^{-1}} \leq |\dot{x}| \leq v_F. \quad (4.2)$$

On each finite parameter interval, positions remain in a bounded Euclidean region. Velocities remain in a compact annulus away from zero. Thus the trajectory in the slit tangent bundle stays in a compact set. The smooth geodesic differential equation extends its solution beyond either finite endpoint. $\square$

This result applies to the prescribed spatial geometry and its affine geodesic parameter.

5 Exact planar scattering relations

Restrict to $d = 2$ and use polar coordinates for $r > 0$. Then

$$\alpha = \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2}, \quad s = \dot{r}/\alpha. \quad (5.1)$$

Rotational symmetry gives conserved angular momentum

$$J = \frac{\partial L}{\partial \dot{\theta}} = F \frac{r^2 \dot{\theta}}{\alpha} B. \quad (5.2)$$

Choose $J > 0$ and define $b = J/v_F$. Angular reflection covers the case $J < 0$. Outside the shell, b equals the usual impact parameter of the incoming Euclidean line. The two first integrals imply

$$\boxed{\frac{b}{r} = \sqrt{1 - s^2} B(r, s).} \quad (5.3)$$

Theorem 5.1 (Scattering and closest approach). *For every $b > 0$, Equation (5.3) has exactly one incoming solution $s_A(r) \in (-1, 0)$ for each $r > b$. The geodesic reaches its unique minimum radius $r = b$ in finite affine parameter. It then follows a Euclidean straight ray. Its total polar-angle change exceeds π if $A(r) > 0$ on a nonempty interval with $r > b$.*

Proof. For fixed r , define $h_r(s) = \sqrt{1 - s^2} B(r, s)$. Direct differentiation gives

$$h'_r(s) = -\frac{s}{\sqrt{1 - s^2}} C(r, s) > 0 \quad (-1 < s < 0). \quad (5.4)$$

The endpoint values are $h_r(-1) = 0$ and $h_r(0) = 1$. Thus $h_r(s) = b/r$ has a unique incoming solution when $r > b$. Reconstruct the velocity through

$$\dot{r} = \frac{v_F s_A}{\phi}, \quad \dot{\theta} = \frac{v_F \sqrt{1 - s_A^2}}{r \phi}. \quad (5.5)$$

These expressions preserve energy and angular momentum. Let $\mathcal{E}_i = d(L_{\dot{x}^i})/d\tau - L_{x^i}$ denote the Euler–Lagrange residuals. Then $\dot{J} = \mathcal{E}_\theta = 0$ and $\dot{E} = \dot{r}\mathcal{E}_r + \dot{\theta}\mathcal{E}_\theta = 0$. Since $\dot{r} \neq 0$ on this branch, both residuals vanish.

At $s = 0$, all derivatives of q vanish, and (5.3) gives $r = b$. Near that point,

$$h_r(s) = 1 - \frac{s^2}{2} + o(s^2), \quad s_A(r) \sim -\sqrt{\frac{2(r - b)}{b}}. \quad (5.6)$$

Since $\dot{r} = v_F s/\phi$, the affine-parameter integral has an integrable square-root endpoint. At the turn, the complete spray equals the Euclidean spray. It gives

$$\ddot{r} = \frac{v_F^2}{b} > 0. \quad (5.7)$$

Smooth uniqueness then gives the outgoing branch. For $s > 0$, all correction derivatives vanish, so that branch is exactly Euclidean and escapes to infinity.

For $s < 0$,

$$q - sq' = e^{-1/s^2} \left(1 - \frac{2}{s^2}\right) < 0. \quad (5.8)$$

Hence $B < 1$ wherever $A > 0$. At fixed $r > b$, Equation (5.3) gives

$$|s_A(r)| < \sqrt{1 - b^2/r^2} \quad (5.9)$$

on each active interval. The angular accumulation per radial interval therefore exceeds its Euclidean value there. The outgoing branch contributes exactly $\pi/2$. The following integral makes the strict excess explicit. $\square$

Define the excess polar angle by $\delta(b) = \Theta_{\text{total}}(b) - \pi$. Since $d\theta/dr = \sqrt{1 - s^2}/(rs)$ on the incoming branch,

$$\delta(b) = \int_b^\infty \left[\frac{\sqrt{1 - s_A(r)^2}}{-s_A(r)} - \frac{b/r}{\sqrt{1 - b^2/r^2}} \right] \frac{dr}{r}. \quad (5.10)$$

The integrand is nonnegative. It is positive on an active interval with $r > b$. If the ray misses the shell, $\delta = 0$. This angle measures deflection within the prescribed spatial geometry.

5.1 Endpoint regularity

The separate angular integrals contain the usual turning-point singularity. Their difference is regular. To verify this, set $t = s^2$ in the implicit equation:

$$\frac{b}{r} = \sqrt{1 - t} \left[1 + A(r)e^{-1/t} \left(1 - \frac{2}{t} \right) \right]. \quad (5.11)$$

Define $k(t) = e^{-1/t}(1 - 2/t)$ for $t > 0$, and set $k(t) = 0$ for $t \leq 0$. This extension is smooth on an open neighborhood of zero. The right-hand derivative with respect to t equals $-1/2$ at $(r, t) = (b, 0)$. The implicit function theorem gives a unique smooth $t(r)$ there. Put $t_0 = 1 - b^2/r^2$. The branch equation gives

$$t - t_0 = (1 - t) \left[2A(r)k(t) + A(r)^2 k(t)^2 \right]. \quad (5.12)$$

The right side vanishes to every order at $r = b$. Consequently, the integrand in (5.10) tends to zero faster than any power of $r - b$. Compact support removes the upper-endpoint issue.

5.2 Linear response and amplitude monotonicity

Set $A(r) = \epsilon \chi(r/\ell)$ and define $u_0(r) = \sqrt{1 - b^2/r^2}$. Differentiation of (5.3) at $\epsilon = 0$ gives

$$\left. \frac{d\delta}{d\epsilon} \right|_0 = \int_b^\infty \frac{b \chi(r/\ell) e^{-1/u_0^2} (2/u_0^2 - 1)}{r^2 u_0^3} dr. \quad (5.13)$$

The exponential controls the endpoint despite the inverse powers. For fixed $b > 0$, this gives $\delta(\epsilon) = \epsilon D(b) + O(\epsilon^2)$. Here $D(b)$ denotes the integral in (5.13).

The sign also holds beyond linear response. Put $k(s) = q(s) - sq'(s) < 0$ on the incoming branch. At fixed r and b ,

$$\frac{\partial s_A}{\partial A} = - \frac{(1 - s_A^2)k(s_A)}{(-s_A)C(r, s_A)} > 0. \quad (5.14)$$

The function $\sqrt{1 - s^2}/(-s)$ increases with s on $(-1, 0)$. Thus a pointwise increase of the nonnegative profile increases the excess angle.

6 Radial trajectories

For $J = 0$, Equation (5.2) and $B > 0$ give $\dot{\theta} = 0$. On the incoming radial branch, $s = -1$ and $q(-1) = e^{-1}$. Constant positive Finsler speed gives

$$\dot{r} = -\frac{v_F}{1 + A(r)/e} < 0 \quad (r > 0). \quad (6.1)$$

Equivalently,

$$\ddot{r} = -\frac{A'(r)/e}{1 + A(r)/e} \dot{r}^2, \quad (6.2)$$

where A' now denotes the radial derivative. Finite smooth A cannot produce a radial reflection at a positive radius. The geodesic passes through the Euclidean neighborhood of the origin. Its Cartesian continuation is smooth there. The coordinate $r = |x|$ changes direction at the origin because it is a distance from that point.

For prescribed inward and outward radial segments between the same two radii, the length difference is

$$L_{\text{in}} - L_{\text{out}} = \frac{1}{e} \int_{r_{\text{low}}}^{r_{\text{high}}} A(r) \, dr. \quad (6.3)$$

The corresponding unit-Finsler-speed parameter intervals have the same difference. These are lengths of specified curves. Global minimization is a separate question.

Orthogonal transformations preserve the metric. Geodesic uniqueness confines each trajectory to the subspace spanned by its initial position and velocity. Every nonradial trajectory therefore reduces to the planar case.

Corollary 6.1 (No trapped geodesics). *Every nonconstant geodesic escapes to infinity at both affine-parameter ends. In particular, the model has no trapped or circular nonconstant geodesics.*

Proof. At any tangent direction with $r > 0$, the radial acceleration equals $v_F^2/r > 0$. This excludes a circular geodesic. Every nonradial incoming geodesic reaches its unique minimum radius and then escapes along a Euclidean ray. Radial geodesics pass through the center and escape. The same branch description extends each geodesic toward its incoming Euclidean asymptote. $\square$

The correction changes directional length and scattering within a globally regular spatial geometry. This result applies to the bounded annular profile and positive definite metric defined here.

7 Reproducible numerical example

The supplied script `verify_model.py` reproduces this section with $\ell = 1$. It uses the profile in (2.4). The scale enters all tabulated impact parameters through b/ℓ . Each table entry is a computed property of the specified profile.

The quadrature first solves (5.3) for its unique incoming root. A second calculation integrates the Cartesian Euler–Lagrange equations directly. It uses the fundamental tensor from (3.7) and analytic spatial derivatives of L . The second calculation does not use the scattering integral to evolve the trajectory.

b/ℓ	$D(b)$	$\delta, \epsilon = 0.2$	$\delta, \epsilon = 0.5$	$\delta, \epsilon = 1$
0.5	0.07831337	0.01699594	0.04912164	0.14041433
1.0	0.28666191	0.06304170	0.17872683	0.38171345
1.5	0.35212777	0.06769015	0.15408895	0.25666823
2.0	0.15877349	0.02785183	0.05884050	0.09433251
2.5	0.00714372	0.00136405	0.00320362	0.00585540
3.0	0	0	0	0

Table 1: Excess angles in radians. $D(b)$ is the derivative at zero amplitude from (5.13).

For five impact parameters at $\epsilon = 1$, the two angle calculations differ by at most 2.4×10^{-12} radians. The maximum sampled Finsler-speed drift is below 1.3×10^{-11} . The maximum sampled angular-momentum drift is below 1.3×10^{-11} in the chosen units. These values describe numerical agreement at the recorded tolerances. Analytical proofs establish the global geometric results.

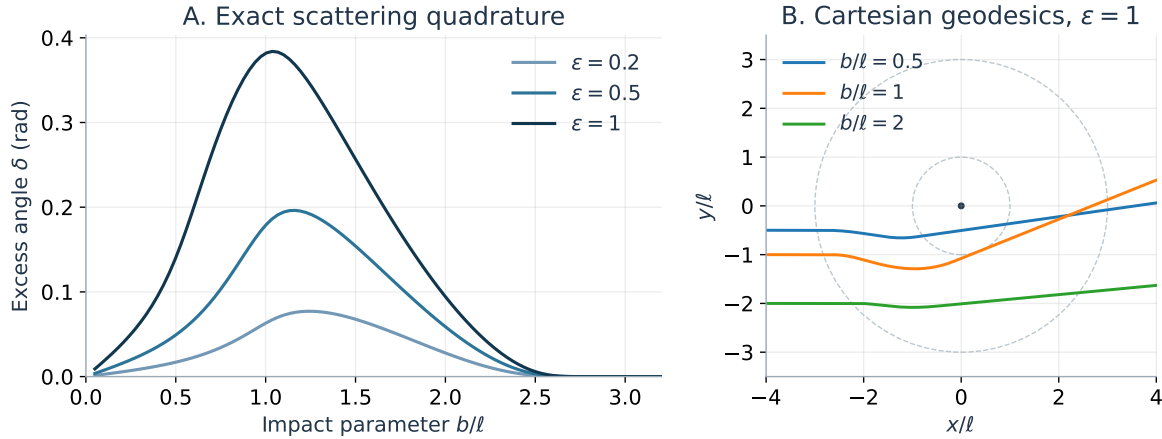


Figure 1: Computed results for the prescribed shell. Panel A shows the excess scattering angle. Panel B shows independent Cartesian geodesic integrations, with motion from left to right. Dashed circles mark $r = \ell$ and $r = 3\ell$. The annular region between them contains the correction. The trajectories curve on their incoming branches and become straight on their outgoing branches.

Additional checks verify the algebra and the local equations:

- Two hundred random samples satisfy positive homogeneity within 4.3×10^{-16} relative error.
- A finite-difference Hessian of L agrees with the analytic tensor within 1.9×10^{-8} at the specified test point.
- A finite-difference spatial-force check agrees within 1.5×10^{-9} at that point.
- A directional grid samples positive tensor eigenvalues for $0 \leq A \leq 1$.
- The critical direction becomes degenerate at A_* and negative above that threshold.

The test point is $x = (2, 0)$, $y = (-0.8, 0.6)$, and $\epsilon = 1$. The finite-difference step is 8×10^{-5} . The script records all sample counts and solver settings.

At $b/\ell = 1$, the residual $[\delta(\epsilon) - \epsilon D(b)]/\epsilon^2$ equals approximately 0.13723, 0.13640, and 0.13597. The amplitudes are 0.04, 0.02, and 0.01, respectively. This behavior agrees with the analytical second-order remainder. Equation (6.3) gives a radial length excess of 0.4439938162ℓ at $\epsilon = 1$ across the full shell.

The recorded environment uses Python 3.12.14, NumPy 2.3.5, SciPy 1.17.0, and Matplotlib 3.10.8. The package includes numerical tables and a machine-readable verification summary. Appendix A gives the reproduction procedure.

8 Scope and further directions

The model describes a prescribed positive definite spatial geometry. Its inward directional penalty changes lengths and nonradial scattering. The proofs establish complete geodesics, exact closest-approach radii, and Euclidean outgoing branches. Radial trajectories cross the shell and center smoothly.

The affine parameter records geodesic progression and has no physical time calibration. The shell scale ℓ and amplitude ϵ remain independent model parameters. The numerical tables use $\ell = 1$. The derivations and supplied code permit direct comparison with other directional profiles.

Curved backgrounds and alternative profiles define extensions of this model. Each extension requires new convexity and trajectory derivations. A physical application also requires equations that determine the background and connect its parameters to measurements.

A Reproduction and numerical details

Use the extracted source directory as the working directory. Run these commands in order:

1. Install the dependencies with `python3 -m pip install -r requirements.txt`.
2. Run `python3 verify_model.py`.
3. Compile `manuscript.tex` with Tectonic.

The numerical script writes the figure and all result tables. The supplied figure also permits compilation without a numerical rerun. The recorded PDF build uses Tectonic 0.17.0 with its TeX Live 2022 bundle. The README supplies the exact command. A complete pdfLaTeX installation also accepts the standard source and requires two passes for cross-references.

The Cartesian solver uses DOP853 with relative tolerance 2×10^{-11} and absolute tolerance 10^{-12} . Its maximum parameter step is 0.05 in the normalized units. Each trajectory starts on $r = 8\ell$, outside the shell. Its initial Cartesian velocity is $(1, 0)$, and its initial Finsler speed is one. The integration ends at the outward crossing of the same circle. The exterior straight segments give the asymptotic deflection directly.

The quadrature uses $r = b/\sin \vartheta$ for $0 < \vartheta < \pi/2$. Put $u = -s_A$, $u_0 = \cos \vartheta$, and

$$K(u) = e^{-1/u^2}(2/u^2 - 1), \quad B = 1 - AK(u). \quad (\text{A.1})$$

The transformed integrand is

$$\frac{u_0}{Bu} - 1 = \frac{AK(u)[2 - AK(u)]}{Bu(u_0 + Bu)}. \quad (\text{A.2})$$

The second expression avoids subtraction of nearly equal terms. The root bracket is $0 \leq u \leq 1$. Very small exponential terms use floating-point cutoffs that lie below the precision of the computed norm. The analytical model retains the exact exponential.

The reproduction archive contains:

- The editable LaTeX manuscript.
- The Python model and verification script.
- The generated figure in PDF and PNG formats.
- Numerical tables in CSV format.
- The complete verification summary in JSON format.
- A dependency specification and reproduction guide.

References

- [1] C. Yu and H. Zhu, “On a new class of Finsler metrics,” *Differential Geometry and its Applications* **29** (2011), 244–254. [doi:10.1016/j.difgeo.2010.12.009](https://doi.org/10.1016/j.difgeo.2010.12.009). Author manuscript: [arXiv:1209.0857](https://arxiv.org/abs/1209.0857).
- [2] X. Mo and L. Zhou, “The curvatures of spherically symmetric Finsler metrics in $\mathbb{R}^n$,” [arXiv:1202.4543v4](https://arxiv.org/abs/1202.4543v4) (2014).